Transmission Line Derivation of 1-D Blackbody Radiation and Thermal Noise.

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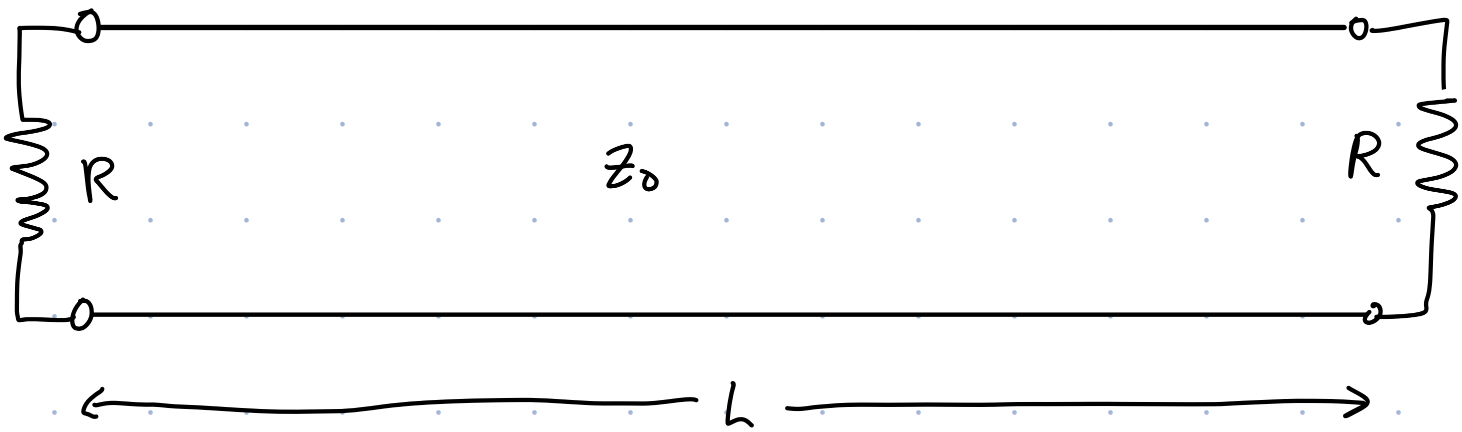
Here, we attempt to use transmission line (TL) formalism to uncover the physics of 1-D blackbody radiation & the thermal noise emitted by a resistor.

In general, TLs can be used to investigate all types of 1-D wave phenomena spanning a wide range of fields in physics.

Consider a TL of characteristic impedance Z_0 that is completely real, terminated at both ends by matched loads $R = Z_0$.



Because the two loads are matched, there are no reflections:

$$\Gamma = \frac{R - Z_0}{R + Z_0} = 0$$

$\therefore$ any wave incident on either R will be completely absorbed.

Quantum mechanically, the EM waves in the TLs have energy $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$

Appealing to statistical mechanics (PHYS 403) @ UBCO, the prob. of mode E_n being excited is given by:

$$P(E_n) = \frac{e^{-E_n/k_B T}}{Z}$$

where $Z = \sum_{n=0}^{\infty} e^{-E_n/k_B T}$ is the partition function. For the harmonic oscillator,

$$Z = \sum_{n=0}^{\infty} e^{-\hbar\omega n/k_B T} e^{-\hbar\omega/2k_B T}$$

$$= e^{-\hbar\omega/2k_B T} \underbrace{\sum_{n=0}^{\infty} e^{-\hbar\omega n/k_B T}}_{\text{geometric series}}$$

$$\frac{1}{1-e^{-\hbar\omega/k_B T}}$$

$$\therefore Z = \frac{e^{-\hbar\omega/2k_B T}}{1-e^{-\hbar\omega/k_B T}}$$

Let's define $\beta = \frac{1}{k_B T}$ such that

Now $$P(E_n) = \frac{e^{-\beta\hbar\omega n}\, \cancel{e^{-\beta\hbar\omega/2}}\, \left(1-e^{-\beta\hbar\omega}\right)}{\cancel{e^{-\beta\hbar\omega/2}}}$$

$$P(E_n) = e^{-\beta\hbar\omega n}\left(1-e^{-\beta\hbar\omega}\right)$$

Check normalization:

$$\sum_{n=0}^{\infty} P(E_n) = 1 = \left(1-e^{-\beta\hbar\omega}\right) \underbrace{\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}}_{\frac{1}{1-e^{-\beta\hbar\omega n}}} \checkmark$$

$$\therefore \langle E \rangle = \sum_{n=0}^{\infty} E_n P(E_n)$$

$$= \sum_{n=0}^{\infty} \hbar\omega\left(n+\frac{1}{2}\right) P(E_n)$$

$$\therefore \langle E \rangle = \frac{\hbar\omega}{2} \underbrace{\sum_{n=0}^{\infty} P(E_n)}_{1} + \hbar\omega\left(1-e^{-\beta\hbar\omega}\right) \underbrace{\sum_{n=0}^{\infty} n e^{-\beta\hbar\omega n}}_{\text{work on this.}}$$

Know that $\displaystyle \sum_{n=0}^{\infty} \underbrace{e^{-\beta\hbar\omega n}}_{\left(e^{-\beta\hbar\omega}\right)^n} = \frac{1}{1-e^{-\beta\hbar\omega}}$

define $r = e^{-\beta\hbar\omega}$ s.t.

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{(usual geometric series)}$$

differential both sides w.r.t. r

$$\frac{d}{dr}\left(\sum_{n=0}^{\infty} r^n\right) = \frac{d}{dr}\left(\frac{1}{1-r}\right)$$

$$\therefore \sum_{n=0}^{\infty} n r^{n-1} = \frac{1}{(1-r)^2}$$

Multiply both sides by r:

$$\therefore \sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2}$$

Now, $$\sum_{n=0}^{\infty} n e^{-\beta\hbar\omega n} = \frac{e^{-\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2}$$

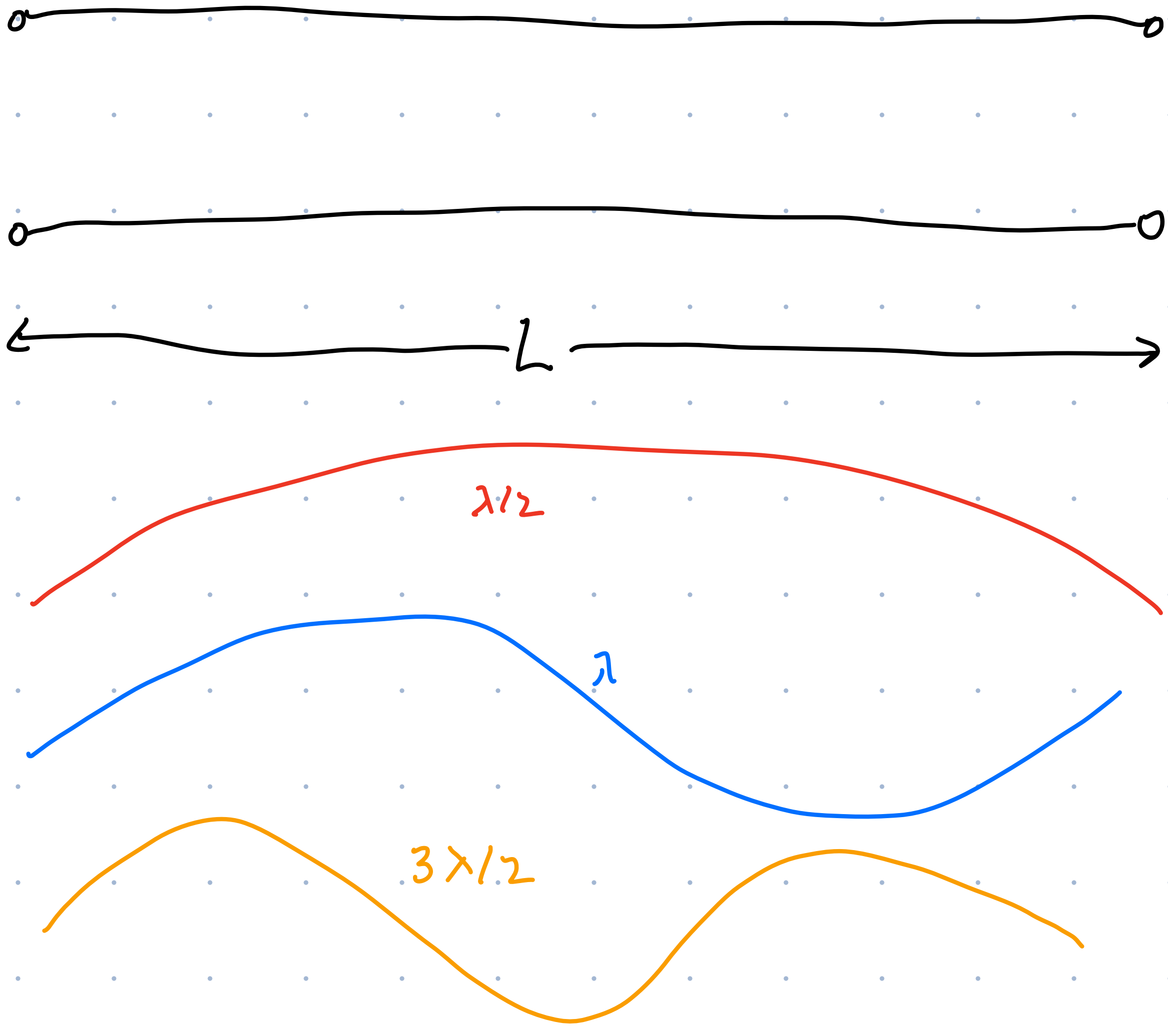
$$\therefore \langle E \rangle = \frac{\hbar\omega}{2} + \hbar\omega \cancel{(1-e^{-\beta\hbar\omega})} \frac{e^{-\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^{\cancel{2}}}$$

$$\boxed{\therefore \langle E \rangle = \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} + \frac{\hbar\omega}{2}}$$

zero point energy.

Average energy of the EM harmonic modes in the TL.

Next, we consider the density of modes or density of states (DOS) in a TL of length L.



$$\therefore \quad L = n\frac{\lambda}{2} \qquad n = 1, 2, \ldots$$

$$\therefore \quad k_n = \frac{2\pi}{\lambda_n} = \frac{2\pi}{\frac{2L}{n}} = \frac{n\pi}{L}$$

$$\therefore \quad \Delta k = k_{n+1} - k_n = \frac{\pi}{L}$$

wave speed

since $c = \frac{\omega}{k}$, $\quad \omega = ck \quad \& \quad \Delta\omega = c\Delta k$

$\therefore \Delta \omega = \dfrac{c\pi}{L}$

$\therefore$ the mode density $\dfrac{1}{\Delta \omega} = \dfrac{L}{\pi c} \equiv g(\omega)$

In terms of energy $g(E) = \dfrac{L}{\pi c \hbar}$ and the number of modes between energy E and $E + \Delta E$ is $g(E)\Delta E = \underbrace{\dfrac{L}{\pi c \hbar}}_{\text{constant.}} \Delta E$.

Therefore, the energy in the TL due to modes w/ energy between E and $E + \Delta E$ is:

$$\underbrace{\langle E \rangle}_{\substack{\text{average}\\ \text{energy per}\\ \text{mode}}} \underbrace{g(E)\Delta E}_{\text{no. of modes.}}$$

∴ In 1-D the energy density is given by:

$$u(E)\,dE = \frac{\langle E\rangle g(E)\,dE}{L}$$

$$= \frac{1}{\pi c\hbar}\left[\frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1} + \frac{\hbar\omega}{2}\right]dE$$

zero pt. energy is usually dropped for temp.-dependent thermodynamic properties.

This is more conventionally written as

$$\boxed{u(\omega)\,d\omega = \frac{\hbar\omega}{\pi c}\,\frac{d\omega}{e^{\hbar\omega/k_BT}-1}}$$

1-D analog of the Planck blackbody spectrum.

Let's now consider the low-freq (long wavelength) limit.

$$e^{\hbar\omega/k_BT} \approx 1 - \frac{\hbar\omega}{k_BT}$$

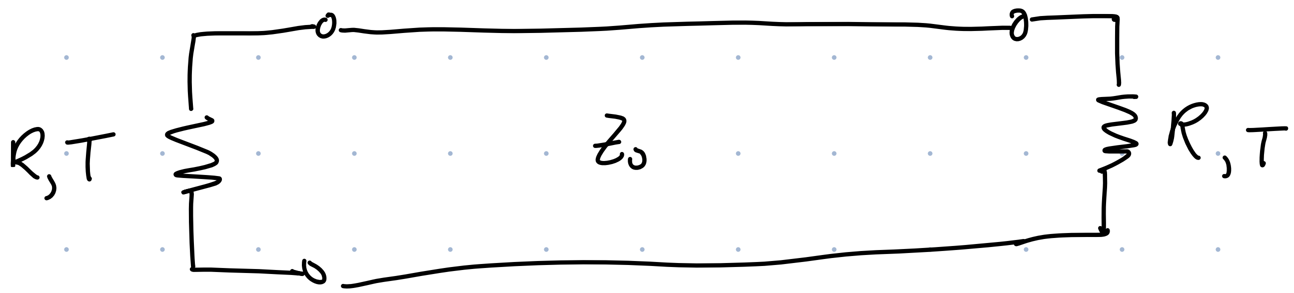
$$\therefore\ u(\omega)\,d\omega \approx \frac{\cancel{\hbar\omega}}{\pi c}\,\frac{d\omega}{\cancel{1} - \frac{\cancel{\hbar\omega}}{k_BT} - \cancel{1}} = \frac{k_BT}{\pi c}\,d\omega$$

Aside.

$$U_{tot} = \int u(\omega)\,d\omega$$

Using the low-frequency limit, $\int_0^\infty \frac{k_B T}{\pi c}\,d\omega$ diverges which is the so-called ultraviolet catrastrophe. It was Planck's original motivation for quantizing the EM field. This low-freq. limit is equivalent to taking $\hbar \to 0$ to recover the classical limit. Classical physics cannot escape the ultraviolet catrastrophe.

Back to the TL.



The origin of energy density $u(\omega)\,d\omega$ is power radiated by the two resistors R at temperature T. Since $R = Z_0$, $\Gamma = 0$ and there are no reflections $\Rightarrow$ all energy incident on R is perfectly absorbed. However, in order to maintain thermal equilibrium (const. T), the resistors must also be radiating energy at the same rate it's absorbing energy $\rightarrow$ no net power loss or gain.

In a TL of length L, the energy in $\Delta\omega$ is

$$u(\omega)\,\Delta\omega = \frac{k_B T}{\pi c}$$

$$u(f)\,\Delta f = \frac{2 k_B T \,\Delta f}{c}$$

$$(\text{Energy density}) \times (\text{wave speed}) \Rightarrow \text{Power}$$

$$\therefore \; P_{net} = 2 k_B T \,\Delta f$$

Half the power is travelling right, due to R on left and the other half is travelling left, due to R on right. (Remember, no net power gain or loss by either resistor in thermal equil.).

$$\boxed{\therefore \; P = k_B T \Delta f}$$

Power radiated by any resistor at temp. T (indep. of R).

Not only is P indep. of R, it is also indep. of f. The resistor radiates power at all frequencies with equal weight over the bandwidth Δf. This uniform power spectral density is called white noise.

The result is valid provided the approximation

$$\frac{\hbar \omega}{k_B T} \ll 1 \text{ is valid.}$$

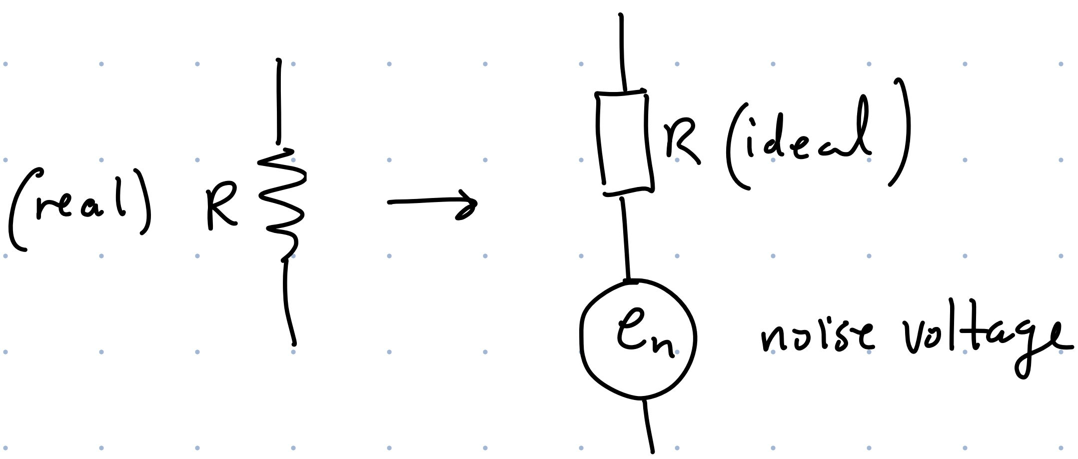
$$\therefore \; f \ll \frac{k_B T}{2\pi \hbar} \quad \text{at room temp } (T \approx 300\,K)$$

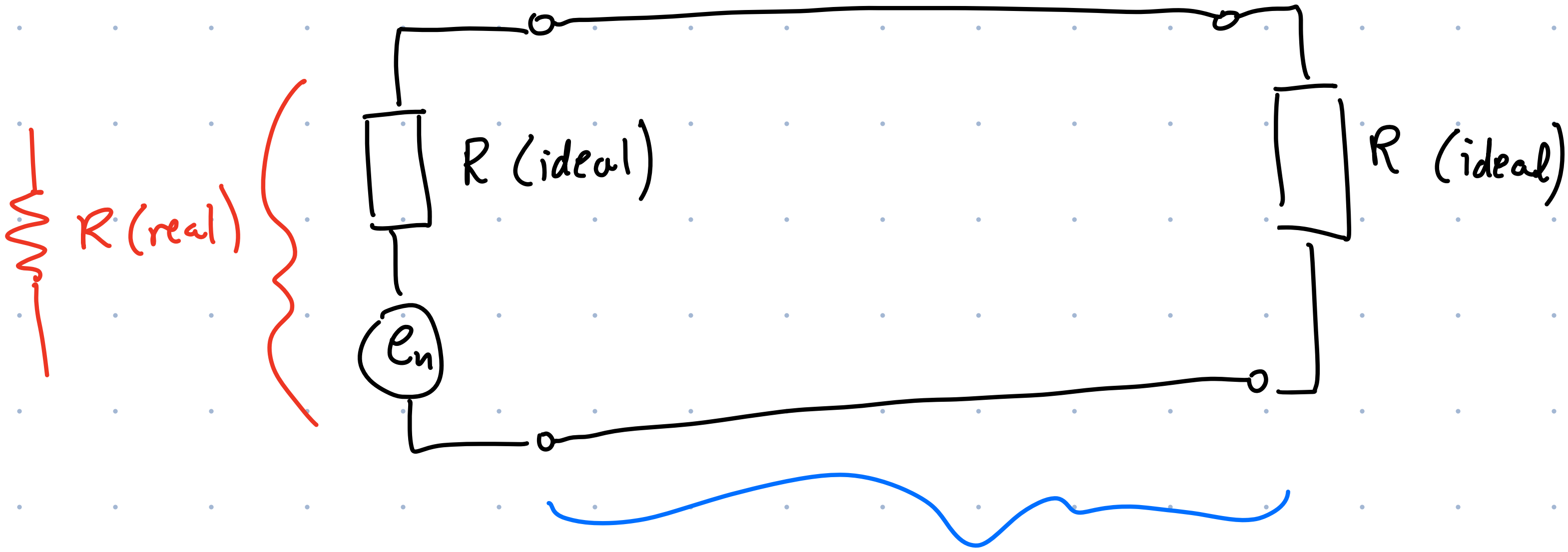
this gives $f \ll 6 \times 10^{12}\,Hz = 6\ THz$.

∴ In the RF/microwave regime (~1 MHz to 100 GHz) and ordinary temperatures, resistors are a very good approximation to **white noise**, although they are truly blackbody radiators.

Often a experimental signal is a voltage, so one would like to convert this noise power to an equivalent noise voltage.

The idea is to return to the TL and terminate one end with a matched "real" noisy resistor and the other end with a matched "ideal" noiseless resistor

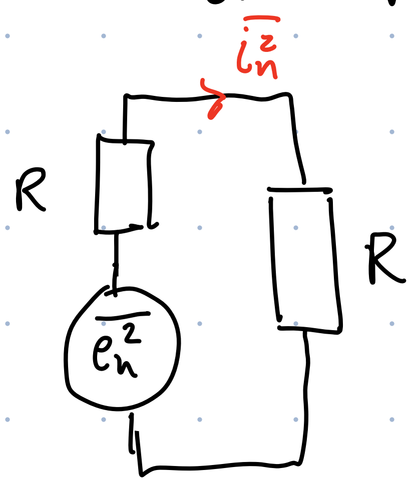




a length of TL terminated by $R = Z_0$.

$\therefore Z_{in} = R$

Equivalent circuit:



We want to calc. the power absorbed by R one the right.

$$\sqrt{\overline{i_n^2}} = \frac{\sqrt{\overline{e_n^2}}}{2R}$$

$$P = \overline{i_n^2}\, R = \left(\frac{\sqrt{\overline{e_n^2}}}{2R}\right)^2 R$$

$$\therefore P = \frac{\overline{e_n^2}}{4R}$$

but we previously found $P = k_B T \Delta f$

$$\therefore \frac{\overline{e_n^2}}{4R} = k_B T \Delta f$$

$$\Rightarrow \boxed{\sqrt{\overline{e_n^2}} = \sqrt{4 k_B T R \Delta f}}$$

Noise voltage of resistor R at temperature T.

This result goes by several different names all describing the same phenomenon:

Thermal Noise / Johnson Noise / Nyquist Noise

Note that $\overline{e_n} = 0$. The noise voltage across a resistor averages to zero.

$\overline{e_n^2}$ is the variance (square of the std. dev.) of the resistor noise voltage.